Name of College –
Deptt – Mathematics
TOPIC – Kummer's Test (Infinite series)
Class – B.Sc II
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Study Material →

Kummer's Test →

Statement – If Σu_n be a series of positive terms and $\Sigma \frac{1}{v_n}$ be a divergent series of positive terms such that

$$\lim_{n\to\infty}\left(v_n \frac{u_n}{u_{n+1}} - v_{n+1}\right) = k$$

then Σu_n is convergent if $k > 0$ and divergent if $k < 0$

Proof :→ Since $\lim_{n\to\infty}\left(v_n \frac{u_n}{u_{n+1}} - v_{n+1}\right) = k$

So there exists a +ve integer m such that

$$k - \varepsilon < \left(v_n \frac{u_n}{u_{n+1}} - v_{n+1}\right) < k + \varepsilon \quad \forall\, n \geq m \quad \text{—(1)}$$

Case I $k > 0$

putting $\epsilon = \frac{1}{2}k$ in the first part of the inequality (1) we get

$$\frac{1}{2}k < \left(v_n \frac{u_n}{u_{n+1}} - v_{n+1}\right) \quad \forall\, n \geq m$$

$$\Rightarrow \frac{1}{2}k\, u_{n+1} < v_n u_n - v_{n+1} u_{n+1} \quad \forall\, n \geq m \text{ as } u_{n+1} > 0$$

Putting $n = m, m+1, m+2 \ldots$ one by one in this inequality, we get

$$\frac{1}{2}k\, u_{m+1} < v_m u_m - v_{m+1} u_{m+1}$$

$$\frac{1}{2}k\, u_{m+2} < v_{m+1} u_{m+1} - v_{m+2} u_{m+2}$$

$$\frac{1}{2}k\, u_{m+2} < v_{m+2} u_{m+2} - v_{m+3} u_{m+3}$$

$$\text{-- -- -- -- -- -- -- -- --}$$

$$\frac{1}{2}k\, u_{l+1} < v_l u_l - v_{l+1} u_{l+1}$$

Adding these we get

$$\frac{1}{2}k\left(u_{m+1} + u_{m+2} + u_{m+3} + \cdots + u_{l+1}\right] < v_m u_m - v_{l+1} u_{l+1}$$

$$\Rightarrow \frac{1}{2}k\left(u_{m+1} + u_{m+2} + \cdots + u_l \ldots\right.$$

Also

$u_1 + u_2 + \cdots + u_l = (u_1 + u_2 + \cdots + u_m) + (u_{m+1} + \cdots + u_l)$

$\therefore$ From (II) we get-

$u_1 + u_2 + \cdots + u_l < (u_1 + u_2 + \cdots + u_m) + \frac{2}{K}(v_m u_m)$

$\forall\ l \geq m.$

Thus $\sum_{n=1}^{l} u_n$ is bounded for all l

and hence $\sum u_n$ is convergent.

Case II $K < 0$

Putting $\varepsilon = -k$ (ε is +ve) in the 2nd part of the inequality (1)

$v_n \frac{u_n}{u_{n+1}} - v_{n+1} < 0 \quad \forall\ n \geq m$

or $v_n u_n - v_{n+1} u_{n+1} < 0 \ \forall\ n \geq m$ as $u_{n+1} > 0$

or $v_n u_n - v_{n+1} u_{n+1} < 0 \quad \forall\ n \geq m$ as $u_{n+1} > 0$

or $v_n u_n < v_{n+1} u_{n+1} \quad \forall\ n \geq m$

Putting $n = m, m+1, m+2, \cdots$ we get-

$v_m u_m < v_{m+1} u_{m+1}$

$v_{m+1} u_{m+1} < v_{m+2} u_{m+2}$

$v_{m+2} u_{m+2} < v_{m+3} u_{m+3}$

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i.e $V_m U_m < V_{m+1} U_{m+1} < V_{m+2} U_{m+2} < \cdots < V_n U_n$

or $V_n U_n > V_m U_m$ whenever $n > m$.

or $U_n > V_m U_m \left(\frac{1}{V_n}\right)$

But $\Sigma \frac{1}{V_n}$ is divergent and $V_m U_m$ is positive as the series ΣU_n $\Sigma \frac{1}{V_n}$ are of positive Terms,

So ΣU_n is divergent series by Comparison Test.

Remarks → Ratio Test and Raabe's Test are particular case of kummer's Test and can be verified by Taking

$$\Sigma \frac{1}{V_n} = 1+1+1 \cdots$$

i.e $V_n = 1$

and $\Sigma \frac{1}{V_n} = \Sigma \frac{1}{n}$

i.e $V_n = n$ respectively.